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MODULI SPACES OF RIEMANN SURFACES OF GENUS TWO WITH LEVEL STRUCTURES. I

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ABSTRACT. The cohomology of modular varieties defined by congruence subgroups of $\mathrm{Sp}_4(\mathbf{Z})$ whose levels lie between 2 and 4 is studied. Using a counting argument and the techniques of zeta functions, the authors completely determine the cohomology of a particular variety of this type.

0. Introduction. In our papers [$\mathbf{LW}_2$, $\mathbf{LW}_3$] we studied the nonsingular moduli space M_2^* of stable curves (=Riemann surfaces) of genus two with level 2 structure. In this paper we start on a program that will, we hope, yield information on the moduli space of curves of genus two with a level Λ structure, where Λ is a "2-primary" congruence subgroup of $\mathrm{PSp}_4(\mathbf{Z})$, i.e. $\Gamma(2) \supset \Lambda \supset \Gamma(2^n)$, for n sufficiently large, where $\Gamma(k)$ is the principal congruence subgroup of level k, i.e. the subgroup consisting of matrices congruent to the identity modulo k. (For such Λ, the index $[\Gamma(2) : \Lambda]$ is a power of 2.)

In part I of this paper we concentrate on a particular subgroup Γ which is midway between $\Gamma(2)$ and $\Gamma(4)$, in the sense that Γ corresponds to a mixed level 2, level 4 structure. (In fact, $[\Gamma(2) : \Gamma(4)] = 2^9$, and $[\Gamma : \Gamma(4)] = 2^3$, so, more optimistically, we are two-thirds of the way from level 2 to level 4.)

By such a structure we mean the following: For a Riemann surface R of genus 2, its Jacobian $J(R)$ is $\mathbf{C}^2/H_1(R : \mathbf{Z})$, an abelian variety of complex dimension 2. The group $H_1(R : \mathbf{Z})$ has a natural symplectic form $\langle , \rangle$ defined on it, coming from the intersection of homology classes, and a level k structure is a choice of symplectic basis for the points of order k on $J(R)$. By a mixed level 2, level 4 structure we mean a level 4 structure on a fixed nonsingular (with respect to $\langle , \rangle$) subspace of rank two, and a level 2 structure on its orthogonal complement.

Indeed, for any Λ there is the idea of a level Λ structure—it is a choice of symplectic basis, where two choices are considered equivalent if the automorphism taking one to the other is an element of Λ.

The moduli space of stable curves which we consider is the Igusa compactification M_Γ^* of the quotient $M_\Gamma = \mathscr{S}_2/\Gamma$, where $\mathscr{S}_k$ is the Siegel space of degree k. By abuse of language, we call $M_\Gamma^* - M_\Gamma$ the boundary ∂ of M_Γ^*. The Siegel space $\mathscr{S}_2$ contains $\mathscr{S}_1 \times \mathscr{S}_1$ as the subspace of diagonal matrices, and we let $\mathscr{S}_2^0$ be $\mathscr{S}_2$ with $\mathscr{S}_1 \times \mathscr{S}_1$ and all of its translates under the action of $\mathrm{PSp}_4(\mathbf{Z})$ deleted. We set $M_\Gamma^0 = \mathscr{S}_2^0/\Gamma$, and we call the closure θ of $M_\Gamma - M_\Gamma^0$ in M_Γ^* the Humbert surface in M_Γ^*.

The space M_Γ^0 parameterizes nonsingular Riemann surfaces of genus two with level Γ structure, and its complement $\partial \cup \theta$ parameterizes stable (in the sense of

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Mumford [$\mathbf{M}_1$, §5]) but singular Riemann surfaces with level Γ structure. (These singularities are described in [$\mathbf{LW}_2$, §8].)

M_Γ^* is a nonsingular projective variety of complex dimension three, and from our description of it below, it is easy to see that it is rational (see (3.10) below.) By [**ArM**], this implies that it is simply-connected and its integral homology is torsion-free.

The main effort of this paper is directed at computing the zeta-function of M_Γ^* at a "good" prime $p \equiv 1$ (4). (Good means that the mod p reduction of M_Γ^* is nonsingular. This must be the case for all but finitely many p, though we cannot identify the bad primes.) This enables us to read off the dimensions of the homology groups $H_i(M_\Gamma^*)$. (For the rest of the introduction we take coefficients in $\mathbf{C}$.) The answer is, from (4.3):

(0.1) $$\dim_{\mathbf{C}} H_i(M_\Gamma^*) = 1, 0, 79, 0, 79, 0, 1 \quad \text{for } i = 0, \dots, 6.$$

In addition to this quantitative result, we have several qualitative results (parts (b) and (c) are proven in §5):

(0.2) (a) $H_i(M_\Gamma^*) = 0$ for i odd.

(b) The map $H_i(\partial \cup \theta) \to H_i(M_\Gamma^*)$ is onto for $i < 6$, i.e. all of the homology of M_Γ^*, except for the fundamental class, comes from the subvarieties parameterizing singular Riemann surfaces.

(c) $H_4(M_\Gamma^*)$ has a basis consisting of algebraic cycles (so in particular, in the Hodge decomposition of $H^*(M_\Gamma^*)$, $H^{p,q} = 0$ unless $p = q$).

Since, for any subgroup $\Gamma \subset \Lambda \subset \Gamma(2)$, $H_i(M_\Lambda^*)$ is the subspace of $H_i(M_\Gamma^*)$ fixed under the action of Λ/Γ, we immediately have the following:

(0.3) For any subgroup Λ, $\Gamma \subset \Lambda \subset \Gamma(2)$, the analogues of (0.2)(a), (b) and (c) hold for M_Λ^*.

Indeed, we have fully determined $H_*(M_\Lambda^*)$ for any such subgroup Λ. The answer will appear in our paper *On certain Siegel modular varieties of genus two and levels above two.*

We have mentioned that $[\Gamma : \Gamma(4)] = 8$, and Γ is generated by $\Gamma(4)$ and elements $A_{22} = (a_{ij})$, $B_{22} = (b_{ij})$, $C_{22} = (c_{ij})$ where $A = B = C =$the identity matrix except that $a_{22} = a_{44} = -1$, $b_{24} = 2$, $c_{42} = 2$. Here we are following the notation of Igusa [**I**]. A_{22} is a strange element. $A_{22}^2 = 1$ and A_{22} is the only torsion element in $\Gamma(2)$.

Part of the interest in these moduli spaces comes from modular forms. In particular, the subgroup $H^{3,0}$ in $H^3(M_\Lambda^*)$ is isomorphic to the space of cusp forms of weight three. In [**I**, Theorem 3] Igusa gives a formula for computing the action of elements of $\mathrm{PSp}_4(\mathbf{Z})$ on modular forms, and it is easy to see from this that if Λ is any subgroup of $\mathrm{PSp}_4(\mathbf{Z})$ containing A_{22}, then there are no modular forms of any odd weight (much less weight three cusp forms) for Λ.

Given the special role that A_{22} plays, we next decided to investigate the subgroup $\tilde{\Gamma}$ generated by $\Gamma(4)$ and the elements B_{22}, C_{22}. Thus $[\tilde{\Gamma} : \Gamma(4)] = 4$, $[\Gamma : \tilde{\Gamma}] = 2$, and in fact $\Gamma/\tilde{\Gamma}$ is generated by A_{22}. Topologically, this means the following: setting $M^* = M_\Gamma^*$, $\tilde{M}^* = M_{\tilde{\Gamma}}^*$ for ease of notation, $\tilde{M}^*$ is a branched two-fold cover of M^* with group of deck transformations $\mathbf{Z}/2 = \{1, A_{22}\}$.

While, by Igusa's formula, there are modular forms of odd weight, there are no cusp forms of weight three, so $H^{3,0}(\tilde{M}^*) = 0$. We have in fact computed $H_*(\tilde{M}^*)$

and the answer is as follows:

$$\dim_{\mathbf{C}} H_i(\tilde{M}^*) = 1, 0, 111, 0, 111, 0, 1 \quad \text{for } i = 0, \dots, 6. \tag{0.4}$$

While the argument to compute $H_*(M^*)$ is via algebraic geometry—computing the zeta function—the argument for computing $H_*(\tilde{M}^*)$ is purely topological. It proceeds as follows: If F is the fixed set of the $\mathbf{Z}_2$ action on $\tilde{M}^*$, there is a generalized Gysin (or Smith) sequence, with $\mathbf{Z}_2$-coefficients

$$\cdots \to H_i(M^*, F) \to H_i(\tilde{M}^*) \to H_i(M^*) \to H_{i-1}(M^*, F) \to \cdots$$

as well as the exact sequence of the pair (M^*, F). By using these sequences, and explicit computations of maps therein (by looking at representing cycles), we may calculate $H_*(\tilde{M}^* : \mathbf{Z}_2)$ and from that derive $H_*(\tilde{M}^* : \mathbf{C})$.

As this method is much different than our method for computing $H_*(M^*)$, and is itself quite involved, we have decided to split up this paper, with our work on $H_*(M^*)$ appearing in part I and our work on $H_*(\tilde{M}^*)$ being deferred to part II.

Our argument in this paper parallels our argument in [**LW**$_2$] for M_2^*, but is considerably more complicated. The first two sections are background. In §1 we define the groups of interest, and recall the structure of the Tits building, which enters into the compactification (recall [**BS**]). Actually, we need a somewhat more elaborate structure, which we call a Tits building with scaffolding. In §2 we describe the structure of certain algebraic curves and surfaces that we will need. The next two sections are the heart of the paper. In §3 we analyze the structure of all of the pieces of M_Γ^*, and in §4 we use this information to calculate the zeta-function. Finally, in §5 we conclude by investigating the question of finding representatives for the homology $H_*(M_\Gamma^*)$.

1. Algebraic preliminaries. First we define some relevant groups and consider their relationships, and then we describe the relevant Tits buildings.

We fix a symplectic inner product $\langle , \rangle$ on $\mathbf{Z}^4$ with matrix $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$.

DEFINITION 1.1. (a) Let $\Gamma(n) = \{M \in \mathrm{PSp}_4(\mathbf{Z}) | M \equiv I \bmod n\}$.

Let

$$\Gamma = \left\{ M \in \mathrm{PSp}_4(\mathbf{Z}) | M \equiv I + 2M' \bmod 4, \quad \text{where } M' \text{ is of the form } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & c & 0 & -a \end{pmatrix} \right\}.$$

(b) Let $\Gamma_1(n) = \{M \in \mathrm{SL}_2(\mathbf{Z}) | M \equiv I \bmod n\}$.

Let

$$\Gamma_1 = \left\{ M \in \mathrm{SL}_2(\mathbf{Z}) | M \equiv I + 2M' \bmod 4, \text{ where } M' \text{ is of the form } \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \right\}.$$

REMARK 1.2. Note we are using the projective group in (a), as it is the projective group which acts effectively on Siegel space, but *not* in (b). The reason for this is to be found in (3.3).

REMARK 1.3. In our announcement of these results [$\mathbf{LW}_0$] we denoted Γ by $\Gamma(2,4)$, but the latter is generally used to denote a different group, so we abandon that notation.

REMARK 1.4. It is easy to see that Γ is the subgroup of $\mathrm{PSp}_4(\mathbf{Z})$ preserving a level 4 structure on the subspace $L = \{(a_1, 0, b_1, 0)\}$ of $\mathbf{Z}^4$ and a level 2 structure on its orthogonal complement $L^\perp = \{(0, a_2, 0, b_2)\}$.

We recall a few more or less standard facts:

LEMMA 1.5. (a) *Let* $\mathrm{sp}_4(\mathbf{Z}/2) = \{M \in \mathrm{Mat}_4(\mathbf{Z}/2) | M = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) \cdot {}^tM \cdot \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)\}$. *Then there exists a homomorphism* $\varphi\colon \Gamma(2^n) \to \mathrm{sp}_4(\mathbf{Z}/2)$ *defined by* $\varphi(X) = (I - X)/2^n \bmod 2^{n+1}$.

(b) *For* $n > 1$ *the following sequence*

$$1 \to \Gamma(2^{n+1}) \to \Gamma(2^n) \xrightarrow{\varphi} \mathrm{sp}_4(\mathbf{Z}/2) \to 1$$

is exact.

(c) *Let* $P\Gamma(2) = \Gamma(2)/(-I)$ *and let* $\mathrm{psp}_4(\mathbf{Z}/2)$ *denote the quotient space of* $\mathrm{sp}_4(\mathbf{Z}/2)$ *modulo the* $\mathbf{Z}/2$*-subspace generated by* $\left(\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right)$. *Then the sequence*

$$1 \to \Gamma(4) \to P\Gamma(2) \xrightarrow{\varphi} \mathrm{psp}_4(\mathbf{Z}/2) \to 1$$

is exact.

(d) $\mathrm{sp}_4(\mathbf{Z}/2)$ *is a* $\mathbf{Z}/2$*-vector space of dimension* 10, *and* $\mathrm{psp}_4(\mathbf{Z}/2)$ *is a* $\mathbf{Z}/2$*-vector space of dimension* 9.

This lemma follows immediately from the identity

$$(I + 2^n M_1)(1 + 2^n M_2) = 1 + 2^n(M_1 + M_2) \mod 2^{n+1} \quad \text{for } n \geq 1.$$

The surjectivity of the map φ is proved by Mumford on p. 207 of [$\mathbf{M}_2$]. The group $\Gamma(4)$ does not contain the central element $\left(\begin{smallmatrix} -I & 0 \\ 0 & -I \end{smallmatrix}\right)$ and so it maps injectively into $P\Gamma(2)$.

The reason for considering $P\Gamma(2)$ is that $\Gamma(2)$ does not operate effectively on the Siegel upper half space S_2. For all practical purposes, we take $P\Gamma(2)$ and consider congruence subgroups which are mapped injectively into $P\Gamma(2)$.

In order to describe our compactifications we must introduce Tits buildings. We refer the reader to [$\mathbf{LW}_1$ or $\mathbf{LW}_3$] (especially the former) for a more extended discussion.

Let $V_n = (\mathbf{Z}/n)^4$. V_n inherits a symplectic form from $\mathbf{Z}^4$, which we still denote by $\langle , \rangle$.

We begin at level 4, i.e. by considering the Tits building of $\mathrm{PSp}_4(V_4)$. The Tits building is a simplicial complex which in our case is 1-dimensional, i.e. it is a graph. There are two sorts of vertices. The first are based lines, that is $\{\pm l \in V_4 | l$ generates a line (a subspace isomorphic to $\mathbf{Z}/4$) in $V_4\}$. The second are based isotropic planes, that is $\{h = \pm l_1 \wedge l_2 \in \Lambda^2(V_4) | h$ is isomorphic to $(\mathbf{Z}/4)^2$ and $\langle , \rangle$ restricted to h is trivial$\}$. There is an edge joining l to h if $l \subset h$.

The Tits building for $\mathrm{PSp}_4(V_4)$ gives a set of instructions for compactifying $\mathscr{S}_2/\Gamma(4)$. One adds a "boundary component" $D_4(l)$ for each line l, and two boundary components $D_4(l)$ and $D_4(l')$ intersect only if l and l' lie in some isotropic plane, i.e. if there is a vertex h to which they are both joined by an edge. We set

$C_4(h) = \bigcup(D(l) \cap D(l'))$ where the union is taken over pairs of lines $l \neq l'$ in h, and call it a cusp component.

Since we wish to begin with M_4^0, we must first do a partial compactification to get to M_4. In order to do this we erect a scaffolding on the Tits building as follows: Let δ be the set of based anisotropic planes in V_4, i.e. $\delta = \{\pm l_1 \wedge l_2 \in \Lambda^2(V_4) | \delta$ is isomorphic to $(\mathbf{Z}/4)^2$ and $\langle , \rangle$ restricted to δ is nonsingular$\}$. Each plane δ has an orthogonal complement $\delta^{\perp}$ and we let Δ be the set of *unordered* pairs $\Delta = \{\delta, \delta^{\perp}\}$. Then the "Humbert components" $H_4(\Delta)$ are parameterized by these pairs Δ, and $H_4(\Delta) \cap D_4(l)$ is nonempty iff $l \in \delta$ or $l \in \delta^{\perp}$.

(Of course, we have not yet described the boundary, cusp, and Humbert components. We will have to do some geometry before we can do so.)

Now it is easy to obtain the Tits building, complete with scaffolding, at level Γ and level 2. The groups $\Gamma/\Gamma(4)$ and $\Gamma(2)/\Gamma(4)$ act on the Tits building and scaffolding at level 4 and we let the structures at level Γ and level 2 be its quotients under these two actions. (At level 2 this amounts to replacing V_4 by V_2.) This description is most convenient for our purposes, as the reader will see, although starting with level 4 appears rather ad hoc. Notice, however, that an equivalent description, for any subgroup Λ of $\mathrm{PSp}_4(\mathbf{Z})$, would be to take as lines $\{\pm l \in \mathbf{Z}^4 | l$ is primitive$\}/\sim$, where $\sim$ is the relation of being equivalent under the action of Λ, and similarly for isotropic and anisotropic planes.

Observe that each line l at level 2 is covered by 8 lines l at level 4, or, equivalently, each $\Gamma(2)/\Gamma(4)$ orbit has 8 elements. (Strictly speaking, we should use different symbols at different levels, but that would lead to a forest of primes, tildes, etc., so we shall forbear.) There are 15 lines at level 2 and so there are 120 at level 4.

Similarly, there are eight isotropic planes h at level 4 over each isotropic plane at level 2 (resp. 16 anisotropic pairs Δ at level 4 over each anisotropic pair at level 2) and there are 15 isotropic planes (resp. 10 anisotropic pairs) at level 2, so there are 120 isotropic planes (resp. 160 anisotropic pairs) at level 4.

There is a duality at each level: Each line at level 2 (level 4) is contained in 3 (6) isotropic planes, and each isotropic plane at level 2 (level 4) contains 3 (6) lines.

Furthermore, each line at level 2 (level 4) is contained in 4 (16) anisotropic pairs, and each such pair contains 6 (12) lines.

It is also true that $D_n(l)$, $C_n(h)$, and $H_n(\Delta)$ are independent of the choice of l, h, and Δ for n=either 2 or 4, so that the whole compactification picture is homogeneous.

This is *false* at level Γ. The different $\Gamma/\Gamma(4)$ orbits of lines, isotropic planes, and anisotropic planes do *not* all have the same cardinality and the corresponding components $D_\Gamma(l)$, $C_\Gamma(h)$, and $H_\Gamma(\Delta)$ are *not* all mutually isomorphic. (The difference arises in $\Gamma(n)$ is normal in $\mathrm{PSp}_4(\mathbf{Z})$ but Γ is not.) Thus, in our analysis of the situation, in §3, we will have to be more careful.

2. Curves and surfaces. In this section we describe some algebraic curves and surfaces that will play a fundamental role in the sequel.

Recall that $\mathscr{S}_i$ denotes Siegel space of degree i, $i = 1, 2$, and $\mathscr{S}_1$ is the upper half plane.

LEMMA 2.1. (a) *The spaces* $\overline{\mathscr{S}_1/\Gamma_1(4)}$, $\overline{\mathscr{S}_1/\Gamma_1}$, *and* $\overline{\mathscr{S}_1/\Gamma_1(2)}$ *are each isomorphic to* $\mathbf{P}^1$, *and have* 6, 4, *and* 3 *cusps respectively.*

(b) *The maps* $\overline{\mathscr{S}_1/\Gamma_1(4)} \to \overline{\mathscr{S}_1/\Gamma_1}$ *and* $\overline{\mathscr{S}_1/\Gamma_1} \to \overline{\mathscr{S}_1/\Gamma_1(2)}$ *are each 2-fold covering maps, branched at two points, the branch points being a subset of the cusps.*

(c) *The composition* $\overline{\mathscr{S}_1/\Gamma_1(4)} \to \overline{\mathscr{S}_1/\Gamma_1(2)}$ *is a 4-fold branched cover with group* $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. *It is branched at the three cusps of* $\overline{\mathscr{S}_1/\Gamma_1(2)}$, *with each cusp being covered by two cusps of* $\overline{\mathscr{S}_1/\Gamma_1(4)}$.

PROOF. The groups $\Gamma_1(4)$, Γ_1, and $\Gamma_1(2)$ are subgroups of $\mathrm{SL}_2(\mathbf{Z})$, which acts ineffectively on $\mathscr{S}_1$; the element $-I$ acts trivially. The images $P\Gamma_1(4)$, $P\Gamma_1$, $P\Gamma_1(2)$ of these groups in $\mathrm{PSL}_2(\mathbf{Z}) = \mathrm{SL}_2(\mathbf{Z})/\pm I$ act effectively, and in fact freely on $\mathscr{S}_1$. It is easy to check that $P\Gamma_1/P\Gamma_1(4) = P\Gamma_1(2)/P\Gamma_1 = \mathbf{Z}_2$, $P\Gamma_1(2)/P\Gamma_1(4) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$, yielding the claims about the covers.

It is also easy to check the claims about the branching—the cover $\overline{\mathscr{S}_1/P\Gamma_1} \to \overline{\mathscr{S}_1/P\Gamma_1(2)}$ is branched at the two cusps 0 and ∞, with the third cusp 1 having two inverse images $+1$ and -1, and these two points are the branch points for the cover $\overline{\mathscr{S}_1/P\Gamma_1(4)} \to \overline{\mathscr{S}_1/P\Gamma_1}$.

Finally, using the classical fact that $\overline{\mathscr{S}_1/\mathrm{PSL}_2(\mathbf{Z})}$ is $\mathbf{P}^1$ with 1 cusp, ∞, (or the almost classical fact that $\overline{\mathscr{S}_1/P\Gamma_1(2)}$ is $\mathbf{P}^1$ with three cusps, 0, 1, and ∞) and computing the Euler characteristics of these branched covers (noting that $\overline{\mathscr{S}_1/P\Gamma_1(2)} \to \overline{\mathscr{S}_1/\mathrm{PSL}_2(\mathbf{Z})}$ has additional branching over elliptic points) shows they all have genus zero, i.e. are all isomorphic to $\mathbf{P}^1$.

(The details of this argument, carried out for an arbitrary subgroup G of $\mathrm{SL}_2(\mathbf{Z})$, may be found in [**Sm**, Chapter 1].)

For our purposes, however, it is important to see that there is a natural identification on $\overline{\mathscr{S}_1/\Gamma_1(2)}$ with $\mathbf{P}^1$ as follows:

$\mathscr{S}_1/\Gamma_1(2)$ is the moduli space of elliptic curves with level 2 structure. An elliptic curve is the Riemann surface of an equation $y^2 = f(x)$, branched at four points a, b, c, d, and an ordering of these points gives a level 2 structure. The curve, with this structure, is uniquely determined by these four points modulo the action of $\mathrm{PGL}_2(\mathbf{C})$, the automorphism group of $\mathbf{P}^1$. An element of $\mathrm{PGL}_2(\mathbf{C})$ is determined by its action on three points, so we may send a to ∞, b to 0, and c to 1, whence d goes to a complex number λ (in fact, to the cross-ratio of a, b, c, d: $\lambda = (d-b)(c-a)/(d-a)(c-b)$). Thus the curve with level 2 structure is given by the equation $y^2 = x(x-1)(x-\lambda)$, the so-called Legendre normal form of the equation, and the moduli space is then clearly $\{\lambda \in \mathbf{P}^1 | \lambda \neq 0, 1, \infty\}$.

Its compactification $\overline{\mathscr{S}_1/\Gamma_1(2)}$ is then clearly $\mathbf{P}^1$ (and the three cusps $0, 1, \infty$ parameterize stable singular elliptic curves with level 2 structure).

PROPOSITION 2.2. $\overline{X} = \overline{\mathscr{S}_1/\Gamma_1(4)}$ *is the Riemann surface of the equation* (*in affine coordinates*)

$$z = f(w) = \left(\frac{w^2+1}{w^2-1}\right)^2.$$

PROOF. By the previous lemma, $\overline{\mathscr{S}_1/\Gamma_1}$ is the two-fold cover of $\overline{\mathscr{S}_1/\Gamma_1(2)} = \mathbf{P}^1$ branched at the points 0 and ∞, i.e. the Riemann surface of $z = y^2$. $\overline{\mathscr{S}_1/\Gamma_1(4)}$

is the two-fold cover of $\overline{\mathscr{S}_1/\Gamma_1}$ branched at the points -1 and 1, i.e. the Riemann surface of $y = ((z+1)/(z-1))^2$. Substitution yields the proposition.

(Observe also that the group of covering translations, $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, is generated by the involutions $w \to -w$ and $w \to 1/w$.)

NOTATION. Throughout this paper X (resp. $\overline{X}$) will denote $\mathscr{S}_1/\Gamma_1(4)$ (resp. $\overline{\mathscr{S}_1/\Gamma_1(4)}$) and f will denote the map in the above proposition.

We now recall the construction of elliptic modular surfaces, originally due to Kodaira [**K**, §8] and much studied by Shioda [**So**], and their close relatives, the Kummer modular surfaces.

Let G be a subgroup of $\mathrm{SL}_2(\mathbf{Z})$ not containing any elliptic elements and $L \subset \mathbf{Z}\oplus\mathbf{Z}$ a G-invariant lattice. Consider the extension of G by L given by the matrix group

$$\overline{G} = \left\{ \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ m & m' & 1 \end{bmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, (m, m') \in L \right\}. \tag{2.3}$$

$\overline{G}$ acts on $\mathscr{S}_1 \times \mathbf{C}$ by

$$(z, w) \mapsto ((zb+d)^{-1}(az+c), (zb+d)^{-1}(w+mz+m')).$$

This covers the action of G on $\mathscr{S}_1$, and so we have a holomorphic fibration

$$\begin{array}{l} D^0 = (\mathscr{S}_1 \times \mathbf{C})/\overline{G} \\ \downarrow \pi \\ B^0 = \mathscr{S}_1/G \end{array}$$

with the base space an open Riemann surface.

In the case G does not contain $-\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ (note that G is a subgroup of $\mathrm{SL}_2(\mathbf{Z})$, *not* $\mathrm{PSL}_2(\mathbf{Z})$), the fiber is an elliptic curve, the quotient of $\mathbf{C}$ by the lattice $\{mz+m' \mid (m, m') \in L\}$.

If G does contain $-\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, this element acts trivially on $\mathscr{S}_1$, but

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

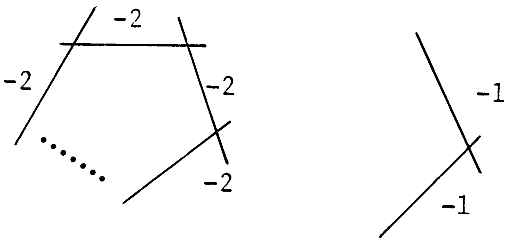
acts nontrivially on $\mathbf{C}$, and the fiber is the quotient of an elliptic curve by this involution, i.e. a Kummer curve ($\mathbf{P}^1$ with four distinguished points, the fixed points of the involution).

In either case, π extends to a map from a nonsingular closed complex surface D to $B = \overline{\mathscr{S}_1/G}$, which we think of as a "singular fibration"; i.e. for a cusp c (a point in $B - B^0$), $\pi^{-1}(c)$ is not an elliptic curve or Kummer curve but rather a certain configuration of projective lines. Various configurations can occur; see [**K**, 6.2] for details.

In the two cases fiber an elliptic curve (topologically a torus) or a Kummer curve (topologically a 2-sphere) we call the closed surface D an elliptic modular surface or a Kummer modular surface respectively.

One important special case is when $G = \Gamma_1(n)$, the principal congruence subgroup of level n in $\mathrm{SL}_2(\mathbf{Z})$ and $L = n\mathbf{Z} \oplus n\mathbf{Z}$. If $n > 2$, G does not contain $-\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and the singular fibers are all "n-gons", i.e. consist of n copies of P^1 with intersections and self-intersections as shown (see [**So**, 4.2]) while if $n = 2$ G contains

$-\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and the situation is also as shown (see [$\mathbf{LW}_3$, 2.1])



(In fact the configuration for $n = 2$ is the quotient of a "2-gon" by an involution.)

3. The structure of M_Γ^*. In this section we determine the structure of all the relevant pieces of M_Γ^*—the cusp components $C_\Gamma(h)$, the boundary components $D_\Gamma(l)$, the Humbert components $H_\Delta(\Gamma)$, and the "interior" M_Γ^0.

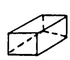
We use the following principle throughout, without further comment: Let G be a group, H and K subgroups of G with $K \subset H \subset N(K)$, where $N(K)$ denotes the normalizer of K. Let G act on a set of objects $\mathscr{A} = \{A_i\}$ by endomorphisms. For $A \in \mathscr{A}$, let $AG = \{Ag | g \in G\}$ be the orbit of A, and $G_A = \{g \in G | Ag = A\}$ be the stabilizer of A. Let $\mathscr{A} \xrightarrow{\pi} \mathscr{A}/K \xrightarrow{\rho} \mathscr{A}/H \xrightarrow{\sigma} \mathscr{A}/N(K)$ be the quotient maps. Then

(i) $\#\{\rho^{-1}(\rho\pi(A))\} = \#\{AH\}/\#\{AK\}$, where $\#$ = cardinality.
(ii) If $(\sigma\rho\pi)(A_1) = (\sigma\rho\pi)(A_2)$, then A_1/K_{A_1} and A_2/K_{A_2} are isomorphic.

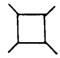
Here we take $G = \mathrm{PSp}_4(\mathbf{Z})/\Gamma(4)$, $H = \Gamma(2)/\Gamma(4)$, $K = \Gamma/\Gamma(4)$, and let $\mathscr{A}$ be some family of components defined at level 4. Note $H \subset N(K)$ as H is abelian. In each case below, every element of G taking h, l, or Δ to another of the same "type" (see below) is in $N(K)$.

DEFINITION 3.1. In the following configurations, each line represents $\mathbf{P}^1$, and each double or triple point the transverse intersection of two or three copies of $\mathbf{P}^1$.

(a) A configuration $\square$ is called a square.

(b) A configuration  is called a cube.

(c) A configuration  is called an open triangle.

(d) A configuration  is called a semicube.

We will write a plane $\pm l_1 \wedge l_2$ as $\binom{l_1}{l_2}$ henceforth. We are about to define "types" of structures at level 2, and we will say that a structure at level Γ covering one of a certain type at level 2 (i.e. which projects to the latter under the action of $\Gamma(2)/\Gamma$) is of the same type.

First, some remarks on compactification. (For a general discussion see [**AMRT**] or [**N**] or [$\mathbf{LW}_1$, §2].) Let Λ be a subgroup of $\mathrm{PSp}_4(\mathbf{Z})$. Recall $M_\Lambda^* = M_\Lambda^0 \cup (\partial \cup \theta)$. (Of course this union must be topologized, and there is a natural topology coming from degenerations of Riemann surfaces.) If we let $\theta^0 = \theta - (\theta \cap \partial)$, then $M_\Lambda^0 = \mathscr{S}_2^0/\Lambda$, $M_\Lambda = M_\Lambda^0 \cup \theta^0 = \mathscr{S}_2/\Lambda$. One component of θ^0 is the quotient of $\mathscr{S}_1 \times \mathscr{S}_1$ by $\Lambda \cap P(\mathrm{SL}_2(\mathbf{Z}) \times\times \mathrm{SL}_2(\mathbf{Z}))$, the latter being the subgroup of $\mathrm{PSp}_4(\mathbf{Z})$

stabilizing $\mathcal{S}_1 \times \mathcal{S}_1$, and the other components will be quotients by the intersection of Λ with conjugates. In the case $\Lambda = \Gamma(n)$, every component of θ^0 will be isomorphic to $\mathcal{S}_1/\Gamma_1(n) \times \mathcal{S}_1/\Gamma_1(n)$, and every component of θ will be isomorphic to $\overline{\mathcal{S}_1/\Gamma_1(n)} \times \overline{\mathcal{S}_1/\Gamma_1(n)}$. The different components of θ are indexed by Δ's, each of which consists a pair of anisotropic planes.

The components of ∂ are indexed by l's, which represent lines in V_4. To describe one, take the line $l = (1,0,0,0)$. Consider $\Lambda \cap P(l)$ where $P(l)$ is the stabilizer of l in $\mathrm{PSp}_4(\mathbf{Z})$,

$$P(l) = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ m & a & 0 & b \\ q & * & 1 & * \\ m' & c & 0 & d \end{pmatrix} \in \mathrm{PSp}_4(\mathbf{Z}) \right\}. \tag{3.2}$$

The entries $*$ above are determined by the requirement that the matrix be symplectic. This condition also forces $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$.

In fact there is a map $\bar{\Phi}$ from $\Lambda \cap P(l)$ to the group $\overline{G}$ of (2.3). This map is the obvious one suggested by the notation. Let Φ be the composition of $\bar{\Phi}$ with the projection $\overline{G} \to G$.

(REMARK 3.3. Note that Φ is well defined as a map from a subgroup of the *projective group* $\mathrm{PSp}_4(\mathbf{Z})$ to a subgroup of the *homogeneous* modular group $\mathrm{SL}_2(\mathbf{Z})$.)

The boundary component $D_\Lambda(1000)$ is the elliptic or Kummer modular surface defined by $\overline{G}$, whose base is $\overline{\mathcal{S}_1/G}$, elliptic if

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \notin \Lambda \cap P(l),$$

Kummer if

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \in \Lambda \cap P(l).$$

(The kernel of $\bar{\Phi}$ consists of all matrices with every variable but $q = 0$. These matrices are in the center of $P(l)$ and indeed act trivially on $\mathcal{S}_1 \times \mathbf{C}$. There is an action of $\Lambda \cap P(l)$ on $\mathcal{S}_1 \times \mathbf{C} \times \mathbf{C}$ which gives a neighborhood of $D_\Lambda(l)$ in M^*_Λ—see [**G**, §1].) If $\Lambda = \Gamma(n)$, $D_\Lambda(l)$ is the elliptic (Kummer) modular surface of level n for $n > 2$ ($n = 2$).

The Humbert and boundary components are all 2-dimensional subvarieties. The cusp components $C_\Lambda(h)$ are unions of 1-dimensional subvarieties, in fact unions of projective lines, exceptional fibers of $D_\Lambda(l)$ for $l \subset h$. Each such projective line is the intersection $D_\Lambda(l) \cap D_\Lambda(l')$, and there are triple points $D_\Lambda(l) \cap D_\Lambda(l') \cap D_\Lambda(l'')$ if $h = \binom{l}{l'} = \binom{l}{l''} = \binom{l'}{l''}$. If each projective line is represented by a line, and each triple point a point, we get a graph. In the case of $\Gamma(n)$, $n > 2$, this graph is a tessellation of a surface (in fact the surface $\overline{\mathcal{S}_1/\Gamma_1(n)}$) by n-gons. For example, if $n = 4$, h contains 6 lines, and the graph is a tessellation of the Riemann sphere by 6 squares, i.e. it is a cube. (If $n = 2$ the graph is the letter Y.) The compactification at level 2 is further discussed in [**LW**$_3$, 8.4] and [**G**, §1].

THEOREM 3.4. *Let*

$$h = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_1' & a_2' & b_1' & b_2' \end{pmatrix}$$

be an isotropic plane at level 2. *If* $a_1b_1' - a_1'b_1 = a_2b_2' - a_2'b_2 = 0$ (*resp.* $= 1$) *we will say that* h *is of type* I (*resp. of type* II).

(a) *There are nine isotropic planes of type* I. *Each is covered by four isotropic planes at level* Γ, *and for each,* $C_\Gamma(h)$ *is a semicube.*

(b) *There are six isotropic planes of type* II. *Each is covered by one isotropic plane at level* Γ, *and for each,* $C_\Gamma(h)$ *is a cube.*

PROOF. Consider an isotropic plane h at level 2.

Suppose $a_1b_1' - a_1'b_1 = 1$. This means that $\langle , \rangle$ restricted to the first and third coordinates is nonsingular, so we may transform those coordinates by an element of $\mathrm{Sp}_2(\mathbf{Z}/2) = \mathrm{SL}_2(\mathbf{Z}/2)$ and asusme that h is of the form

$$\begin{pmatrix} 1 & a_2 & 0 & b_2 \\ 0 & a_2' & 1 & b_2' \end{pmatrix}.$$

Now $a_2b_2' - a_2'b_2 = 1$, i.e. the submatrix formed by the second and fourth coordinates is an element of $\mathrm{SL}_2(\mathbf{Z}/2)$, which has cardinality 6.

If $a_ib_i' - a_i'b_i = 0$, $i = 1, 2$, then the submatrices formed by the first and third columns, and by the second and fourth, each have rank 1, but h has rank 2. Thus, after perhaps a change of basis, we may assume that the first of these submatrices is either $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and the second is either $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, so there are 9 possibilities.

(Note that by our specific choice of basis we have removed the ambiguity in sign.)

Each of the 15 isotropic planes at level 2 is covered by 8 planes at level 4. They are:

$$\text{Type I:} \quad \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} + 2\begin{pmatrix} 0 & a & 0 & b \\ 0 & c & 0 & -a \end{pmatrix},$$

$$\text{Type II:} \quad \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & b & c \end{pmatrix},$$

where $(a, b, c) \in (\mathbf{Z}/2)^3$.

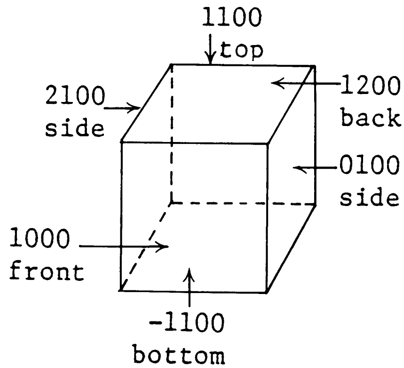
In the case of type I, $\Gamma/\Gamma(4)$ acts freely on the planes at level 4 lying over a given plane at level 2 it is essentially the action of $\Gamma/\Gamma(4)$ on itself—so a $\Gamma/\Gamma(4)$ orbit has size 8 and such a plane at level 2 is covered by 1 plane at level Γ.

In the case of type II, a $\Gamma/\Gamma(4)$ orbit has size 2, as the action of $\Gamma/\Gamma(4)$ preserves the values of a and b but may alter the value of c. Thus such a plane at level 2 is covered by 4 planes at level Γ.

Now we must determine the structure of $C_\Gamma(h)$.

At level 4, as we have noted, each $C_4(h)$ is a cube. Since $\Gamma/\Gamma(h)$ acts freely on these cubes for lines of type I, the quotient $C_\Gamma(h)$ is also a cube. As for type II, the stabilizer of a plane h in $\Gamma/\Gamma(4)$ has order 4 (as a $\Gamma/\Gamma(4)$ orbit is of size 2). We consider the case $h = \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right)$.

At level 4, $C_4(h)$ is the cube with faces the lines as shown:

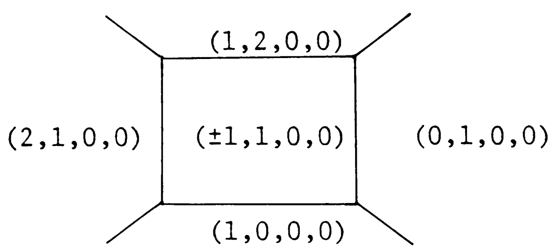


The stabilizer of h in $\Gamma/\Gamma(4)$ is generated by two elements,

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & 2 & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

The first of these acts trivially on the configuration, while the second switches the top and bottom faces but leaves the sides invariant. Thus the quotient is a semicube (with faces as labelled):

(3.5)



THEOREM 3.6. *Let $l = \pm(a_1, a_2, b_1, b_2)$ be a line at level* 2.
If $(a_2, b_2) = (0,0)$ we will say l is of type I.
If $(a_1, b_1) = (0,0)$ we will say l is of type II.
Otherwise, l is of type III.

(a) *There are three lines l of type* I. *Each is covered by eight lines at the Γ level. For each, $D_\Gamma(l)$ is a Kummer modular surface with base $\overline{\mathscr{S}_1/\Gamma_1(2)}$ and each of the three singular fibers an open triangle.*

(b) *There are three lines l of type* II. *Each is covered by four lines at the Γ level. For each, $D_\Gamma(l)$ is a Kummer modular surface with base $\overline{\mathscr{S}_1/\Gamma_1(4)}$ and each of the six singular fibers an open triangle.*

(c) *There are nine lines l of type* III. *Each is covered by two lines at the Γ level. For each, $D_\Gamma(l)$ is an elliptic modular surface with base $\overline{\mathscr{S}_1/\Gamma_1}$ and each of the four singular fibers a square.*

PROOF. Clearly the number of lines of each type at level 2 is as claimed. Each is covered by eight lines at level 4.

As $\Gamma/\Gamma(4)$ only affects the second and fourth coordinates, it acts trivially on lines at level 4 covering a line of type I, so there are eight lines at level Γ covering such a line at level 2.

Indeed, the action of $\Gamma/\Gamma(4)$ is to vary a_2 and b_2 arbitrarily, so there are four possibilities. However, in the case of a line of type II, $a_1 \equiv b_1 \equiv 0 \bmod 2$ so there is an ambiguity of sign, and a $\Gamma/\Gamma(4)$ orbit is of size two, and so there are four lines at the Γ level covering a line of type II at level 2. On the other hand, if a_1 and b_1 are not both even, they specify a sign and there is no ambiguity, so a $\Gamma/\Gamma(4)$ orbit is of size four, and there are two lines at the Γ level covering each line of type III at level 2.

As a line of type I we take $l = (1000)$. In this case we have described $P(l)$ in (3.2). Using the notation of that section, it is clear that the group $G = \Phi(\Gamma \cap P(l))$ is $\Gamma_1(2)$. Hence $D_\Gamma(l)$ is a Kummer modular surface over $\overline{\mathscr{S}_1/\Gamma_1(2)}$. The general fiber is P^1 and there are three singular fibers, whose identification we defer.

Given another line l, let $g \in \mathrm{Sp}_4(\mathbf{Z})$ be an element with $lg = (1000)$. Then $P(l) = g^{-1}P(1000)g$, so $\Gamma \cap P(l) = \Gamma \cap g^{-1}P(1000)g$.

Consider the line $l = (0100)$ of type II. Then we may take $g = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$, which gives

$$P(0100) = \left\{ \pm \begin{pmatrix} a & m & b & 0 \\ 0 & 1 & 0 & 0 \\ c & m' & d & 0 \\ * & q & * & 1 \end{pmatrix} \right\}.$$

Note that we must take $\pm$ as, using the fact that we are in the projective group, we have normalized our matrices by requiring that the entry in the upper left-hand corner be $+1$. Then

$$\Gamma \cap P(0100)/\Gamma(4) \cap P(0100) = \left\{ \begin{pmatrix} 1 & & & \\ & \pm 1 & & \\ & & 1 & \\ & q & & \pm 1 \end{pmatrix} \bmod 4,\ q = 0, 2 \right\}.$$

Then if $G = \Phi(\Gamma \cap P(0100))$, G is an extension of $\Gamma_1(4)$ by $-I$, so $\overline{\mathscr{S}_1/G} = \overline{\mathscr{S}_1/\Gamma_1(4)}$. Note, however, that $\overline{G} = \bar{\Phi}(\Gamma \cap P(0100))$ contains

$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix},$$

so $D_\Gamma(l)$ is a Kummer modular surface over $\overline{\mathscr{S}_1/\Gamma_1(4)}$, with six singular fibers, yet to be identified.

Now consider the line $l = (1100)$ of type III. One can again check that

$$\Gamma \cap P(1100)/\Gamma(4) \cap P(1100) = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & c & & 1 \end{pmatrix} \bmod 4,\ c = 0, 2 \right\}.$$

Hence G in this case is the group Γ_1, and $\overline{G}$ does not contain

$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix},$$

so here $D_\Gamma(l)$ is an elliptic modular surface with base $\overline{\mathscr{S}_1/\Gamma_1}$ and four singular fibers, yet to be identified.

Now for the singular fibers: We can identify these easily by noting that every singular fiber is contained in a cusp component, and we have already analyzed the cusp components.

For example, consider the cusp component $C_\Gamma(h)$ for $h = \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right)$. This is the case given in (3.5). Note therein that the central square is a singular fiber in $D_\Gamma(1100)$, while the bottom open triangle is a singular fiber in $D_\Gamma(1000)$, and the right-hand open triangle is a singular fiber in $D_\Gamma(0100)$. Thus each of the boundary components has a singular fiber as claimed in the theorem. We are not quite finished, because we have claimed that all of the singular fibers in each of the $D_\Gamma(l)$'s are as claimed, and a priori the singular fibers in a $D_\Gamma(l)$ need not be mutually isomorphic. Thus to complete the proof we must check some additional cusp components, but this is now routine, and this checking yields the theorem.

THEOREM 3.7. *Let $\Delta = \{\delta, \delta^\perp\}$ be an anisotropic pair of level* 2. *Without loss of generality, we may assume that*

$$\delta = \begin{pmatrix} 1 & a_2 & 0 & b_2 \\ 0 & a_2' & 1 & b_2' \end{pmatrix}.$$

If $(a_2, b_2) = (a_2', b_2') = (0,0)$ we will say Δ is of type I. *Otherwise Δ is of type* II.

(a) *There is one isotropic plane Δ of type* I. *It is covered by* 16 *isotropic planes at the Γ level. For each, $H_\Gamma(\Delta)$ is isomorphic to $\overline{\mathscr{S}_1/\Gamma_1(4)} \times \overline{\mathscr{S}_1/\Gamma_1(2)}$.*

(b) *There are nine isotropic planes Δ of type* II. *Each is covered by* 4 *isotropic planes at the Γ level. For each, $H_\Gamma(\Delta)$ is isomorphic to $\overline{\mathscr{S}_1/\Gamma_1(4)} \times \overline{\mathscr{S}_1/\Gamma_1}$.*

REMARK 3.8. By (2.1), in either case $H_\Gamma(\Delta)$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$.

PROOF. If

$$\delta = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_1' & a_2' & b_1' & b_2' \end{pmatrix}$$

then, since $\langle , \rangle$ restricted to δ is nonsingular, $1 = (a_1 b_1' - a_1' b_1) + (a_2 b_2' - a_2' b_2)$. Interchanging δ and δ' if necessary, we may assume the first of these summands is equal to 1, and then by a change of basis (as in (3.4)) assume

$$\begin{pmatrix} a_1 & b_1 \\ a_1' & b_1' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

There are 10 pairs at level 2. Clearly one is of type I and the rest of type II. Each is covered by 16 pairs at level 4 (a_2, b_2, a_2', and b_2' may be 0 or 2). As $\Gamma/\Gamma(4)$ only affects the second and fourth coordinates, it acts trivially on the anisotropic pairs at level 4 covering Δ of type I, so there are 16 such at level Γ covering this one at level 2. It is easy to check that a $\Gamma/\Gamma(4)$ orbit on the anisotropic pairs at level 4 covering Δ of type II has size 4, so there are four such at level Γ covering each of these at level 2.

The rest of the argument resembles the proof of (3.6).

Again we analyze one representative of each type.

If $\delta = \left(\begin{smallmatrix}1&0&0&0\\0&0&1&0\end{smallmatrix}\right)$, its stabilizer $P(\delta) = P(\delta^{\perp})$ in $\mathrm{PSp}_4(\mathbf{Z})$ is of the form

$$\begin{pmatrix} a & & b & \\ & a' & & b' \\ c & & d & \\ & c' & & d' \end{pmatrix}$$

with the primed and unprimed matrices in $\mathrm{SL}_2(\mathbf{Z})$. (There is a harmless ambiguity of sign here.) This copy of $\mathrm{SL}_2(\mathbf{Z}) \times \mathrm{SL}_2(\mathbf{Z})$ acts on $\mathscr{S}_1 \times \mathscr{S}_1$ in the usual way, with quotient $H_\Gamma(\Delta) \cap \theta^0$. The Humbert component $H_\Gamma(\Delta)$ is the quotient of the usual action on $\overline{\mathscr{S}_1} \times \overline{\mathscr{S}}_1$.

Clearly $\Gamma \cap P(\delta)$ is $\Gamma_1(4) \times \Gamma_1(2) \subset \mathrm{SL}_2(\mathbf{Z}) \times \mathrm{SL}_2(\mathbf{Z})$, so $H_\Gamma(\Delta) = \overline{\mathscr{S}_1/\Gamma_1(4)} \times \overline{\mathscr{S}_1/\Gamma_1(2)}$.

If $\delta = \left(\begin{smallmatrix}1&1&0&0\\0&0&1&0\end{smallmatrix}\right)$ it is easy to see that $\Gamma \cap P(\delta)$ is conjugate to $\Gamma_1(4) \times \Gamma_1$, so $H_\Gamma(\Delta) = \overline{\mathscr{S}_1/\Gamma_1(4)} \times \overline{\mathscr{S}_1/\Gamma_1}$.

THEOREM 3.9. *Let M_Γ^0 be the moduli space of nonsingular curves of genus two with mixed level 2, level 4 structure. Then $M_\Gamma^0 = X \times X \times X - \tilde{\Delta}$ where $X = \mathscr{S}_1/\Gamma_1(4)$ and $\tilde{\Delta} = \{(x_1, x_2, x_3) | f(x_i)$ not all distinct$\}$.*

PROOF. We begin with some general remarks, following Mumford [$\mathbf{M}_2$, IIIa.8].

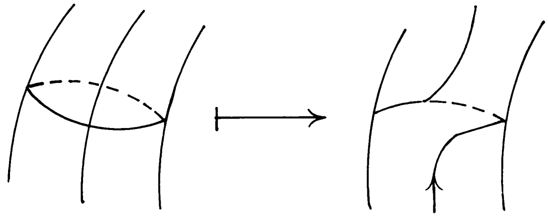
Consider the moduli space of hyperelliptic curves of genus g with ordered Weierstrass points. By this we mean the following: Such a curve is a 2-fold cover of $\mathbf{P}^1$ branched at $2g+2$ points, and the Weierstrass points are the inverse images of these points. Denote this moduli space by M; let $*$ be a basepoint in M and R_* the Riemann surface it parameterizes. Pick a symplectic basis $e_1, \ldots, e_g$, $f_1, \ldots, f_g$ for $H_1(M : \mathbf{Z})$, i.e. a basis with $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$, $\langle e_i, f_j \rangle = \delta_{ij}$.

Consider a closed path in M, beginning and ending at $*$. Traversing this path gives a family of Riemann surfaces R_t, $t \in [0,1]$. Now $R_0 = R_*$, and $R_1 = R_*$ as well, but in traversing this path the basis for H_1 will in general have changed to a new symplectic basis $e_1', \ldots, f_g'$. The map $\{e_i, f_i\} \to \{e_i', f_i'\}$ is a symplectic change of basis, i.e. an element of $\mathrm{Sp}_{2g}(\mathbf{Z})$. As this latter group is discrete, homotopic maps give the same automorphism, and so we obtain a map $\sigma\colon \pi_1(M, *) \to \mathrm{PSp}_{2g}(\mathbf{Z})$.

Since a hyperelliptic curve with ordered Weierstrass points is determined by its branching data, uniquely up to the action of $\mathrm{PGL}_2(\mathbf{C})$, the automorphism group of $\mathbf{P}^1$, we have that

$$M = \{\text{distinct points in } (\mathbf{P}^1)^{2g+2}\}/\mathrm{PGL}_2(\mathbf{C})$$

and it is easy to see that $\pi_1(M, *)$ is generated by curves c which move 1 of these branch points in $\mathbf{P}^1$ but leave all the others fixed. If we let $\sigma_c = \sigma(c)$, then for such a curve σ_c is the automorphism of $H_1(R_*)$ given by $\sigma_c\colon x \mapsto x + 2\langle x, c\rangle c$, i.e. by performing a Dehn twist (see [**Li**]) along two copies of c (the factor of 2 arises as R_* is a 2-fold cover of $\mathbf{P}^1$, so c lifts to two curves in R_*.) A Dehn twist is given by the following (we illustrate its effect on the curve x, which is transformed into x'):



Finally, the image of σ is precisely the principal congruence subgroup of level 2 of $\mathrm{PSp}_{2g}(\mathbf{Z})$.

Now we specialize to our situation. We have in this case $M = M_2^0$, the space of [**LW**$_2$, **LW**$_3$].

For any homomorphism ψ of $\Gamma(2)/\Gamma(4)$ into a group G consider the composition

$$\Psi\colon \pi_1(M_2^0) \xrightarrow{\sigma} \Gamma(2) \xrightarrow{\text{mod } 4} \Gamma(2)/\Gamma(4) \xrightarrow{\psi} G.$$

There is the covering M_Λ^0 of M_2^0 corresponding to the subgroup $\Lambda = \mathrm{Ker}(\Psi)$ of $\pi_1(M_2^0)$, and this is a covering between level 2 and level 4 in the sense that we have a sequence of covers $M_4^0 \to M_\Gamma^0 \to M_2^0$.

For example, if ψ is the trivial map, $M_\Lambda^0 = M_2^0$, and if ψ is the identity map, $M_\Lambda^0 = M_4^0$. We shall be interested in the case that ψ is the quotient map $\psi_0\colon \Gamma(2)/\Gamma(4) \to \Gamma(2)/\Gamma$. Note a closed loop c in M_2^0 lifts to a closed loop in M_Γ^0 iff $\sigma_c \in \Gamma$.

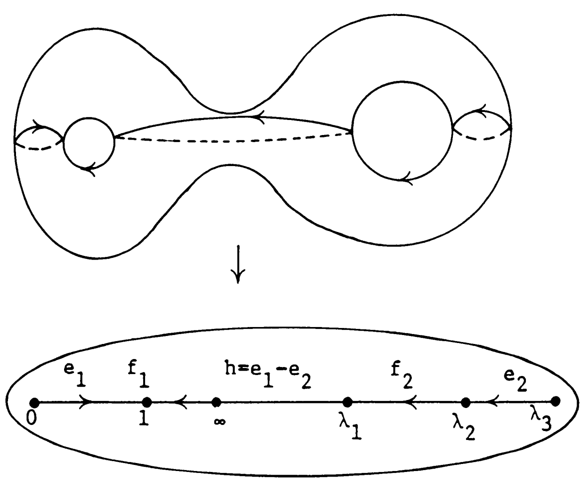
We have identified (see [**LW**$_3$]) M_2^0 as

$$\begin{aligned} M_2^0 &= (\mathscr{S}_1/\Gamma_1(2))^3 - \Delta = \{(x_1, x_2, x_3) \in (\mathbf{P}^1 - \{0, 1, \infty\})^3 | x_i \text{ all distinct}\} \\ &= \{(x_1, x_2, x_3) \in (\mathbf{C} - \{0, 1\})^3 | x_i \text{ all distinct}\}. \end{aligned}$$

Let the Riemann surface R_* be represented by the equation

$$y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3),$$

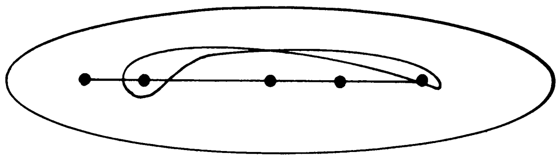
the double cover of $\mathbf{P}^1$ branched at the *ordered* set of points $\infty, 0, 1, \lambda_1, \lambda_2, \lambda_3$. Pick the standard basis e_i, f_i, $i = 1, 2$, of R_*. It is represented by the lifts of the branch cuts in the diagram below, and we have also drawn the branch cut lifting to the curve joining the third and fourth branch points.



From the description of M_2^0 above, it is clear that $\pi_1(M_2^0)$ is generated by the following nine elements:

$$p_i = \text{loop around } x_i = 0 \text{ in } i\text{th factor } i = 1, 2, 3,$$
$$q_i = \text{loop around } x_i = 1 \text{ in } i\text{th factor } i = 1, 2, 3,$$
$$n_i = \text{loop around } x_j = x_k, \ \{i, j, k\} = \{1, 2, 3\}.$$

(Note that r_i=loop around ∞ in ith factor is not independent, as $p_i q_i r_i = 1$.) Thus, for example, q_3 is represented by the loop which is $f_1+h+f_2+e_2 = e_1+f_1+f_2$.



The nine generators map to Dehn twists along the following curves:

$$\begin{array}{lll} p_1 \to -e_2 + f_1, & p_2 \to -e_2 + f_1 + f_2, & p_3 \to f_1 + f_2, \\ q_1 \to e_1 - e_2 + f_1, & q_2 \to e_1 - e_2 + f_1 + f_2, & q_3 \to e_1 + f_1 + f_2, \\ n_1 \to e_2, & n_2 \to e_2 + f_2, & n_3 \to f_2. \end{array}$$

In particular,

$$\sigma_{n_1} = I + \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & -2 & & 0 \end{pmatrix}, \qquad \sigma_{n_2} = I + \begin{pmatrix} 0 & & & \\ & 2 & & 2 \\ & & 0 & \\ & 2 & & -2 \end{pmatrix},$$

$$\sigma_{n_3} = I + \begin{pmatrix} 0 & & & \\ & 0 & & 2 \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

so that $\sigma_{n_1}, \sigma_{n_2}, \sigma_{n_3}$ generate $\Gamma/\Gamma(4)$. Hence

$$\Psi_0(n_1) = \Psi_0(n_2) = \Psi_0(n_3) = 0; \qquad \{\Psi_0(p_i), \Psi_0(q_i)\} \text{ generate } \Gamma(2)/\Gamma.$$

Thus M_Γ^0 will be a cover of M_2^0 such that each factor of the projection has branching of order 2 (since $\Gamma(2)/\Gamma$ has exponent 2) at the points 0,1, and ∞ (since $\Psi_0(r_1) \neq 0$ as well) and nowhere else. However, by (2.2), this is precisely the description of $X = \mathscr{S}_1/\Gamma_1(4)$ and so

$$M_\Gamma^0 = (\mathscr{S}_1/\Gamma_1(4))^3 - f^{-1}(\Delta) = X \times X \times X - \tilde{\Delta}$$

as claimed.

COROLLARY 3.10. *M_Γ^* is rational.*

PROOF. (Compare [**LW**$_3$, 4.1.1].) M_Γ^* contains M_Γ^0 as a Zariski open set. However, we have just identified M_Γ^0 as a Zariski open set in $\overline{X}^3 = (\mathbf{P}^1)^3$ which is rational, so M_Γ^* is rational.

4. The zeta-function. In this section we arrive at our main result, the computation of the zeta-function of M_Γ^*. First we must deal with the boundary components.

PROPOSITION 4.1. *Let p be a good prime.*

(a) *If l is of type* I, *the zeta-function of $D_\Gamma(l)$ is given by*

$$\varsigma(s) = \frac{1}{(1-s)(1-ps)^8(1-p^2s)}.$$

(b) *If l is of type* II, *the zeta-function of $D_\Gamma(l)$ is given by*

$$\varsigma(s) = \frac{1}{(1-s)(1-ps)^{14}(1-p^2s)}.$$

(c) *If l is of type* III, *the zeta-function of $D_\Gamma(l)$ is given by*

$$\varsigma(s) = \frac{1}{(1-s)(1-ps)^{14}(1-p^2s)}.$$

PROOF. For an elliptic or Kummer modular surface D, a singular fibration over a base B, $H_1(D) = H_1(B)$ (see [**So**, §2] or [**LW**$_1$, §3] for the elliptic case, the Kummer case is immediate as all fibers, both nonsingular and singular, are simply-connected).

For all three types here, B is $\mathbf{P}^1$, by (2.1). (In fact, it is the case here that $D_\Gamma(l)$ is always simply-connected.)

From Theorem 3.6 it is easy to compute the Euler characteristic of $D_\Gamma(l)$, and in cases I, II, and III it is 10, 16, and 16 respectively, so H_2 has rank 8, 14, and 14 respectively.

To complete the proof it then suffices to show that this is also the rank of the Neron-Severi group in each case, i.e. that all of H_2 is generated by algebraic cycles, as this implies that the zeta-function has the desired form [**Tt**].

In each case the following constitutes a basis for $H_2(D_\Gamma(l) : \mathbf{Q})$:

(i) All but one of the projective lines in each exceptional fiber.

(ii) The remaining projective line in any one of the exceptional fibers.

(iii) One of the sections (i.e. $D_\Gamma(l) \cap H_\Gamma(\Delta)$ for any Δ where the intersection is nonempty).

(There is of course a lot of choice here. The union of the exceptional fibers and sections maps onto $H_2(D_\Gamma(l) : \mathbf{Q})$ but there is a big kernel.)

In each of the three cases we have the right number of elements to form a basis for H_2, and to show that they are indeed independent we show that their self-intersection matrix is nonsingular (as in [**So**, 1.1]) as follows:

Choose a section s as in (iii). This section passes through one of the projective lines in each exceptional fiber, so choose all of the remaining ones as in (i). The general (nonsingular) fiber f is homologous to the sum of the projective lines in any exceptional fiber, so choosing a projective line as in (ii) is the same as adding f to the claimed basis. Now if $\cdot$ denotes intersection number, $f \cdot f = 0$ (as f may be pushed off itself to an adjacent fiber), $f \cdot s = 1$, $f \cdot p_i = s \cdot p_i = 0$. Hence the intersection matrix is a block matrix (where $* = s \cdot s$)

$$\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & * & \\ \hline & & B \end{array}\right).$$

This is nonsingular if B is. But B itself decomposes into a number of blocks, each containing the projective lines in a single singular fiber chosen in (i). If p is a projective line in a square, $p \cdot p = -2$, and if p is a projective line in an open triangle, $p \cdot p = -1$ or -2 according as p is one of the two "outside" or the "middle" line (compare the computation in [**LW**$_3$, 2.3.1]).

In either case, regardless of the choice made in (i), each block is readily seen to have nonzero determinant, so B is nonsingular.

THEOREM 4.2. *Let p be a good prime congruent to* 1 *modulo* 4. *Then the zeta-function of M_Γ^* is given by*

$$\varsigma(s) = \frac{1}{(1-p^3s)(1-p^2s)^{79}(1-ps)^{79}(1-s)}.$$

COROLLARY 4.3. *M_Γ^* is simply connected and the integral cohomology $H^i(M_\Gamma^*)$ is free abelian of rank* $1, 0, 79, 0, 79, 0, 1$ *for* $i = 0, \ldots, 6$.

PROOF OF COROLLARY. The Weil conjectures enable us to read off the dimensions of the rational cohomology groups from the zeta-function. The sharper statement in the corollary then follows from the rationality of M_Γ^*, shown in (3.4), by a result of Artin and Mumford [**ArM**].

REMARK 4.4. By a good prime p we mean one at which the mod p reduction of M_Γ^* is nonsingular. There are only finitely many bad primes, but we cannot decide which they are.

PROOF OF THEOREM. Fix a good prime p congruent to 1 modulo 4.

First note that for any variety M, if M is the disjoint union $M_1 \cup M_2$, then $\varsigma_M(s) = \varsigma_{M_1}(s)\varsigma_{M_2}(s)$. This follows from the interpretation of the zeta function as $\varsigma(s) = \exp Z(t)$, $t = p^{-s}$, where $Z(t)$ "counts points"—a point in M is either in M_1 or M_2. More generally, if $M = M_1 \cup M_2$, $M_0 = M_1 \cap M_2$, then $\varsigma_M(s) = \varsigma_{M_1}(s)\varsigma_{M_2}(s)/\varsigma_{M_0}(s)$ as in forming the numerator, points in M_0 are counted twice, and dividing by $\varsigma_{M_0}(s)$ corrects for this. Similarly, there is a correction to be made if $M = M_1 \cup M_2 \cup M_3$ and there are triple points.

We compute the zeta function of M_Γ^* as follows: M_Γ^* is the disjoint union of two pieces: M_Γ^0 and $M_\Gamma^* - M_\Gamma^0$. The first piece is the moduli space of nonsingular curves, and we compute its zeta-function by counting points. The second piece is $\partial \cup \theta$, the union of the boundary and the Humbert surface, and we compute its zeta-function by using Proposition 4.1.

By Theorem 3.9, $M_\Gamma^0 = X \times X \times X - \tilde{\Delta}$. By (2.1) and (2.2), X is the Riemann surface of $f(w) = ((w^2+1)/(w^2-1))^2$ and is itself isomorphic to $\mathbf{P}^1$.

Now we pass to characteristic p and count the points in M_Γ^0. Suppose we are counting the points in M_Γ^0 defined over $k = GF(p^n)$, the finite field with p^n elements.

Consider a point $(x_1, x_2, x_3) \in M_\Gamma^0$. For x_1 we may choose any $x \in \mathbf{P}^1(k)$ with $f(x) \neq 0, 1, \infty$. Now $f^{-1}(\infty) = \pm 1$, $f^{-1}(1) = 0, \infty$, *and since* $p \equiv 1(4)$, $f^{-1}(1)$ *is two points*, the two solutions of $w^2 = -1$ in GF(p^n).

Thus

$$\begin{aligned} \operatorname{card}(\{x_1\}) &= \operatorname{card}(X - f^{-1}(\{0, 1, \infty\})) \\ &= \operatorname{card}(\mathbf{P}^1(k) - 6 \text{ points}) = p^n - 5. \end{aligned}$$

For x_2 we may choose any $x \in \mathbf{P}^1(k)$ with $f(x) \neq 0, 1, \infty$, or $f(x_1)$, i.e. $x_2 \in X - f^{-1}\{0, 1, \infty, f(x_1)\}$. Now $f^{-1}(f(x_1))$ is four points $(x_1, -x_1, 1/x_1, -1/x_1)$ so $\operatorname{card}(\{x_2\}) = \operatorname{card}(\mathbf{P}^1 - 10 \text{ points}) = p^n - 9$. Similarly,

$$\operatorname{card}(\{x_3\}) = \operatorname{card}(\mathbf{P}^1 - 14 \text{ points}) = p^n - 13.$$

Hence $\nu_n(M_\Gamma^0)$, the cardinality of M_Γ^0 as a variety over $\mathrm{GF}(p^n)$, is given by $\nu_n(M_\Gamma^0) = (p^n - 5)(p^n - 9)(p^n - 13)$, and its zeta function is given by

$$\begin{aligned} \varsigma(s) &= \exp \sum_{n=1}^{\infty} \nu_n(M_\Gamma^0) \frac{s^n}{n} \\ &= \exp \sum_{n=1}^{\infty} (p^{3n} - 27p^{2n} + 227p^n - 585) \frac{s^n}{n} \\ &= \frac{(1-p^2)^{27}(1-s)^{585}}{(1-p^3 s)(1-ps)^{227}}. \end{aligned} \tag{4.5}$$

Now for $\partial \cup \theta$: By (3.8), each Humbert surface, of either type I or type II, is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ and so has zeta-function $\varsigma(s) = 1/((1-p^2 s)(1-ps)^2(1-s))$, and there are 52 Humbert surfaces.

By (3.6), there are 24 boundary components of type I, 12 of type II, and 18 of type III, and their zeta-functions are given by (4.1).

Thus, before correcting for the fact that we do not have a disjoint union, the contribution of $\partial \cup \theta$ to the zeta-function of M_Γ^* is

$$\begin{aligned} &\left[\frac{1}{(1-p^2 s)(1-ps)^2(1-s)}\right]^{52} \cdot \left[\frac{1}{(1-p^2 s)(1-ps)^8(1-s)}\right]^{24} \\ &\cdot \left[\frac{1}{(1-p^2 s)(1-ps)^{14}(1-s)}\right]^{12} \cdot \left[\frac{1}{(1-p^2 s)(1-ps)^{14}(1-s)}\right]^{18}. \end{aligned} \tag{4.6}$$

Now we must correct for the multiple points. They arise in two ways—as intersections $H_\Gamma(\Delta) \cap D_\Gamma(l)$ and $D_\Gamma(l) \cap D_\Gamma(l')$. In either case the intersection is $\mathbf{P}^1$, with zeta-function $1/(1-ps)(1-s)$.

An intersection $H_\Gamma(\Delta) \cap D_\Gamma(l)$ lies in $H_\Gamma(\Delta)$ as either $\{\text{cusp}\} \times \mathbf{P}^1$ or $\mathbf{P}^1 \times \{\text{cusp}\}$. There are 16 planes Δ of type I, each containing six cusps in the first factor and three in the second, for a total of 9 copies of $\mathbf{P}^1$. There are 36 planes Δ of type II, each containing six cusps in the first factor and four in the second, for a total of 10 copies of $\mathbf{P}^1$.

An intersection $D_\Gamma(l) \cap D_\Gamma(l')$ lies in a cusp component $C_\Gamma(h)$. There are 36 cusp components of type I, each of which is a semicube, i.e. containing 8 copies of $\mathbf{P}^1$, and 6 of type II, each of which is a cube, i.e. containing 12 copies of $\mathbf{P}^1$.

Thus between the two types of intersections we have $16.9 + 36.10 + 36.8 + 6.12 = 864$ copies of $\mathbf{P}^1$, so we must correct (4.6) by *dividing* by

$$[1/(1-ps)(1-s)]^{864}. \tag{4.7}$$

Now for the triple points: If we have a triple point $X \cap Y \cap Z$, we have so far counted it three times (once each for X, Y, and Z) and subtracted it out three times (once each for $X \cap Y$, $Y \cap Z$, and $X \cap Z$) for a net count of zero. Thus we must correct our computation by now counting the triple points, i.e. correct our zeta function by multiplying by the zeta-function of the triple points.

Each triple point is an isolated point, with zeta-function $1/(1-s)$, so we need only count these points. They arise in two ways—as $H_\Gamma(\Delta) \cap D_\Gamma(l) \cap D_\Gamma(l')$ and $D_\Gamma(l) \cap D_\Gamma(l') \cap D_\Gamma(l'')$. Points of the first kind are points in $H_\Gamma(\Delta)$ of the form (c, c') with each of c and c' a cusp. There are 16 $H_\Gamma(\Delta)$ of type I, each containing 6.3=18 such points, and 36 of type II, each containing 6.4=24 such points. Points of the second kind are vertices in $C_\Gamma(h)$. There are 36 of these of type I, each with 4 vertices, and 6 of type II, each with 8 vertices. Thus there are a total of $16.18 + 36.24 + 36.4 + 6.8 = 1344$ triple points, so we must *multiply* our expression for the zeta-function of $\partial \cup \theta$ by

$$[1/(1-s)]^{1344} \tag{4.8}$$

Hence, assembling (4.6), (4.7), and (4.8), the contribution of $\partial \cup \theta$ to the zeta-function of M^*_Γ is

$$(1-ps)^{148}/(1-p^2s)^{106}(1-s)^{586}. \tag{4.9}$$

Then, multiplying (4.5) and (4.9), we find that the zeta-function of M^*_Γ is given by the expression in the statement of the theorem.

5. Representing homology classes. In this section we consider the question of finding geometric representatives for the homology of M^*_Γ.

THEOREM 5.1. *The inclusion $\partial \cup \theta \to M^*_\Gamma$ induces epimorphisms on rational homology in dimensions less than six.*

PROOF. Recall that $M^0_\Gamma = M^*_\Gamma - (\partial \cup \theta) = X \times X \times X - \tilde{\Delta}$.
Thus we may describe M^0_Γ as the total space of a "2-stage" fibration

$$\begin{array}{ccc} F_2 & \longrightarrow & M^0_\Gamma \\ & & \downarrow \\ F_1 & \longrightarrow & E_1 \\ & & \downarrow \\ & & X \end{array}$$

where a point in M^0_Γ has coordinates (x_1, x_2, x_3), with

$$x_1 \in X = \mathbf{P}^1 - 6 \text{ points},$$
$$x_2 \in F_1 = \{z \in X | f(z) \neq f(x_1)\} = \mathbf{P}^1 - 10 \text{ points},$$
$$x_3 \in F_2 = \{z \in X | f(z) \neq f(x_1), f(z) \neq f(x_2)\} = \mathbf{P}^1 - 14 \text{ points}.$$

Thus X, F_1, and F_2 have the homotopy type of a wedge of 5, 9, and 13 circles respectively. Computing the cohomology of M^0_Γ via spectral sequences we see immediately that $H^1(M^0_\Gamma : \mathbf{Q})$ has rank ≤ 27 (in fact equal, as we will see below) and $H^i(M^0_\Gamma : \mathbf{Q}) = 0$ for $i > 3$.

Now consider the exact sequences (with rational coefficients):

$$H^1(M^0_\Gamma) \to H_4(\partial \cup \theta) \xrightarrow{i_4} H_4(M^*_\Gamma),$$
$$H_2(\partial \cup \theta) \xrightarrow{i_2} H_2(M^*_\Gamma) \to H^4(M^0_\Gamma)$$

which are part of the long exact homology sequence of the pair $(M_\Gamma^*, \partial \cup \theta)$ combined with the Alexander duality isomorphisms $H^{6-i}(M_\Gamma^0) = H_i(M_\Gamma^*, \partial \cup \theta)$.

The space $\partial \cup \theta$ is the union of the 54 boundary components $D_\Gamma(l)$ and the 52 Humbert surfaces $H_\Gamma(\Delta)$. While the union is not disjoint, the intersection of any two of these has complex codimension 1 (=real codimension 2) so $H_4(\partial \cup \theta)$ has rank $106 = 54 + 52$. We have shown that $H_4(M_\Gamma^*)$ has rank 79 and $H^1(M_\Gamma^0)$ has rank ≤ 27, so the map i_4 above must be onto. Also, $H^4(M_\Gamma^0) = 0$, so i_2 is certainly onto.

COROLLARY 5.2. *$H_4(M_\Gamma^* : \mathbf{Q})$ has a basis represented by algebraic cycles.*

PROOF. $H_4(\partial \cup \theta)$ is generated by the fundamental classes of the $D_\Gamma(l)$ and $H_\Gamma(\Delta)$.

REMARK 5.3. By the proof of (4.1), $H_2(D_\Gamma(l))$ has a basis represented by algebraic cycles, for every l, and this is certainly also true for $H_\Gamma(\Delta)$, as each of these is $\mathbf{P}^1 \times \mathbf{P}^1$. We are morally certain that $H_2(M_\Gamma^* : \mathbf{Q})$ has a basis represented by algebraic cycles, but as $\bigoplus_l H_2(D_\Gamma(l)) \oplus \bigoplus_\Delta H_2(H_\Gamma(\Delta)) \to H_2(\partial \cup \theta)$ is not onto, this moral certainly is unfortunately not a proof.

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